Evolutionary Algorithms and Matroid Optimization Problems *

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ABSTRACT

We analyze the performance of evolutionary algorithms on various matroid optimization problems that encompass a vast number of efficiently solvable as well as NP-hard combinatorial optimization problems (including many well-known examples such as minimum spanning tree and maximum bipartite matching). We obtain very promising bounds on the expected running time and quality of the computed solution. Our results establish a better theoretical understanding of why randomized search heuristics yield empirically good results for many real-world optimization problems.

Categories and Subject Descriptors

G.2.1 [**Combinatorics**]: Combinatorial algorithms; F.2.2 [**Nonnumerical Algorithms and Problems**]: Computations on discrete structures

General Terms

Theory, Algorithms, Performance

Keywords

evolutionary algorithms, matroids, minimum weight basis, matroid intersection, randomized search heuristics

1. INTRODUCTION

Motivation. While evolutionary algorithms are known to work (empirically) well for many optimization problems in practice, a satisfying and rigorous mathematical analysis of their performance is one of the main challenges in the area of genetic and evolutionary computing. Interesting results have been obtained for some important, isolated optimization problems but a general theoretical explanation of the behavior of evolutionary algorithms is still missing. The aim of this paper is to make progress in this direction. We study the performance of evolutionary algorithms on a very general class of combinatorial optimization problems and obtain promising results on the running time and quality of the computed solutions. A summary of our results together with an overview of the complexity of the considered optimization problems can be found in Table 1.

Independence Systems and Matroids. Matroid theory provides a framework in which a substantial class of problems in combinatorial optimization can be studied from a unified perspective. Matroids form a special class of *independence systems* that are given by a finite set E and a family of subsets $\mathcal{F} \subseteq 2^E$ such that $\mathcal{F}$ is closed under subsets. The subsets contained in $\mathcal{F}$ are called *independent* and a maximal independent subset is called a *basis* of the independence system. A precise definition of matroids along with some important classical results is given in Section 2.

Many combinatorial optimization problems can be formulated as follows: Given an independence system on a weighted set E, find a basis of minimum (or maximum) weight. We mention as an example the problem of finding a stable set with maximum weight in a given graph with weights on the nodes (here, the underlying independence system is *not* a matroid).

A famous result by Rado [19], Gale [7], and Edmonds [4] states that an independence system is a matroid if and only if the greedy algorithm computes a minimum weight basis for arbitrary weights on the elements of E. This algorithmic characterization of matroids highlights their relevance in the area of efficient algorithms and combinatorial optimization.

Examples of Matroids and First Results. *Graphic matroids* are an important example of matroids where E is the edge set of a graph $G = (V, E)$ and a subset of edges is independent if it does not contain a circuit. If the given graph G is connected, the problem of finding a minimum weight basis of the corresponding graphic matroid is the minimum spanning tree problem. In the context of evolutionary algorithms, Raidl and Julstrom [20] analyze different encodings for the minimum spanning tree problem and propose to work with so-called *edge sets*. Neumann and Wegener [16] study evolutionary algorithms for the minimum spanning tree problem. They prove that two specific evolutionary algorithms, (1+1) EA and RLS, compute a minimum spanning tree in expected polynomial time, i.e., $O(|E|^2(\log|V| + \log w_{\max}))$, where $w_{\max}$ denotes the maximum weight of any edge; moreover, they establish a lower bound of $\Omega(|E|^2 \log|V|)$. In Section 3 we generalize this result to arbitrary matroids (see also Table 1).

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problem	efficient algorithm known	(1+1) EA and RLS	lower bound on (1+1) EA and RLS
minimum weight basis	$O(\|E\| \log \|E\|)$	$O(\|E\|^2(\log r(E) + \log w_{\max}))$	$\Omega(\|E\|^2 \log r(E))$ [16]
unweighted matroid intersection	$O(\|E\|^{2.5})$	$(1-\varepsilon)$-approximation in $O(\|E\|^{2\lceil 1/\varepsilon \rceil})$	exponential [9]
weighted matroid intersection	$O(\|E\|^{2.5}(\log \|E\| + \log w_{\max}))$	1/2-approximation in $O(\|E\|^4(\log \|E\| + \log w_{\max}))$	exponential [9]
intersection of $p \geq 3$ matroids	NP-hard [8]	$1/p$-approximation in $O(\|E\|^{p+2}(\log \|E\| + \log w_{\max}))$	exponential

Table 1: A summary of results on various matroid optimization problems. The problems are described in the first column. The second column gives the running time of known efficient algorithms. The third column describes the results for (1+1) EA and (modifications of) RLS obtained in this paper. The last column gives lower bounds on the running time of (1+1) EA and RLS for obtaining an *optimal* solution.

Another important example of matroids are *linear matroids* where E is a set of vectors and a subset is independent if the vectors contained in it are linearly independent. Linear matroids occur, for example, in the minimum cycle basis problem where the task is to find a minimum weight basis of the cycle space of a given graph with weights on the edges. This problem is an important building block in various real-world optimization problems such as, for example, in electrical networks, structural engineering, chemistry and biochemistry, and in periodic timetabling; see, e.g., [15] for details.

Matroid Intersection. Matroids have even more algorithmic power than just that of the greedy method. Edmonds [3, 14] observed that also the *(weighted) matroid intersection problem* can be solved efficiently. That is, a maximum weight common independent set in two matroids can be found in strongly polynomial time. The matroid intersection problem has applications in many settings such as, for example, edge connectivity [5], survivable network design [1], constrained minimum spanning trees [12], and multicast network codes [11].

The most prominent example of an optimization problem that can be formulated as a matroid intersection problem is the maximum weight matching problem in bipartite graphs. Giel and Wegener [9, 10] analyze evolutionary algorithms for the maximum matching problem (with unit weights). They show that (1+1) EA and RLS are (randomized) polynomial-time approximation schemes with an expected runtime of $O(|E|^{2\lceil 1/\epsilon \rceil})$; moreover they construct a class of bipartite graphs for which these algorithms need an exponential expected running time until they find an optimal solution. Motivated by these results we prove in Section 4 that (1+1) EA and RLS are polynomial-time approximation schemes for the matroid intersection problem with unit weights (see also Table 1).

Our result for the unweighted matroid intersection problem cannot be generalized easily to the weighted case and also from the viewpoint of efficient algorithms it is known that the weighted version of the problem is somewhat harder than the unweighted problem. In Section 5 we prove that (1+1) EA and a slightly modified version of RLS are $\frac{1}{2}$-approximation algorithms for the weighted matroid intersection problem (see also Table 1).

NP-Hard Problems. The problem of finding a maximum-size common independent set in three or more matroids is NP-hard as finding a Hamiltonian circuit in a directed graph is a special case [13]. On the other hand it is known that any independence system can be represented as an intersection of finitely many matroids and a vast number of combinatorial optimization problem falls into this category. In Section 6 we prove that (1+1) EA and (an appropriately modified version of) RLS are $\frac{1}{p}$-approximation algorithms for the NP-hard problem to find a maximum-weight independent set in the intersection of $p \geq 3$ matroids (see also Table 1). Note that the approximation ratio of $\frac{1}{p}$ is the same as that of the greedy algorithm [14].

2. PRELIMINARIES

We study in this paper the behavior of two simple evolutionary algorithms, namely (1+1) EA and randomized local search (RLS), which is sometimes also called *local* (1+1) EA. Both algorithms operate on bitstrings of fixed length and differ in the mutation operator. Initially, a bitstring $s \in \{0,1\}^n$ is chosen randomly (whenever we speak of randomness, we mean uniform randomness). The mutation operators of (1+1) EA and RLS are defined as follows:

(1+1) EA: Obtain the bitstring s' by flipping each bit of s independently of the other bits with probability $1/n$.

RLS: Choose $b \in \{0,1\}$ randomly. If $b = 0$, choose $i \in \{1, \ldots, n\}$ randomly and obtain s' by flipping the i-th bit of s. If $b = 1$, choose $(i,j) \in \{(k,l) \mid 1 \leq k < l \leq n\}$ randomly and obtain s' by flipping the i-th and j-th bit of s.

The current search point s is replaced by s' if the fitness value of s' is better than or equal to that of s. This procedure repeats until a stopping criterion is met. In this paper we analyze the expected number of steps until we reach a certain fitness value corresponding to the problem solution.

We shall now give the definition of matroids as well as some well-known properties used in this paper. See [14], [18], or [21], for a more detailed discussion.

Definition 2.1. *Let E be a finite set and $\mathcal{F} \subseteq 2^E$. The pair $M = (E, \mathcal{F})$ is called a* matroid *if*

(i) $\emptyset \in \mathcal{F}$,
(ii) $\forall X \subseteq Y \in \mathcal{F}: \ X \in \mathcal{F}$, *and*
(iii) $\forall X, Y \in \mathcal{F}, |X| > |Y| : \exists x \in X \setminus Y$ *with* $Y \cup \{x\} \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called *independent*, the elements of $2^E \setminus \mathcal{F}$ are called *dependent*. The maximal independent sets are called *bases of* M, the minimal dependent sets are called *circuits*. For $X \subseteq E$, a maximal independent subset of X is called a *basis of* X. The *rank* $r(X)$ *of* X is the maximal

cardinality of a basis of X, $r(X) := \max\{|Y| \mid Y \subseteq X, Y \in \mathcal{F}\}$.

PROPOSITION 2.2. *Let $(E, \mathcal{F})$ be a matroid. Then*
(i) for $X \subseteq E$, all bases of X have the same cardinality;
(ii) for all bases B_1, B_2 of M, $x \in B_1 \setminus B_2$ there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a basis of M.

PROPOSITION 2.3. *Let $(E, \mathcal{F})$ be a matroid and $r(\cdot)$ its rank function. Then*
(i) $r(\emptyset) = 0$,
(ii) $\forall X \subseteq E : r(X) \leq |X|$,
(iii) $\forall X \subseteq E : r(X) = |X| \Leftrightarrow X \in \mathcal{F}$,
(iv) $\forall X, Y \subseteq E, X \subseteq Y : r(X) \leq r(Y)$.

For $X \in \mathcal{F}$ and $y \in E$ such that $X \cup \{y\} \notin \mathcal{F}$, we use $C(X, y)$ to denote the unique circuit in $X \cup \{y\}$. If $X \cup \{y\} \in \mathcal{F}$, we define $C(X, y) := \emptyset$.

3. MINIMUM WEIGHT BASIS

The results presented in this section generalize and are motivated by the corresponding results for the minimum spanning tree problem in [16]. We consider the following problem. Given a matroid $M = (E, \mathcal{F})$ and a weight function $w : E \to \mathbb{N}$, find a basis $B \subseteq E$ of M of minimum weight. The weight of a subset of E is defined as the sum of the weights of its elements. The weight of an optimal solution is denoted by w_{OPT}.

We assume that the set $\mathcal{F}$ of independent subsets is implicitly given by a rank oracle that for any set $X \subseteq E$ computes its rank $r(X)$. Note that a rank oracle is polynomially equivalent to an independence oracle [14].

The search space equals $S = \{0,1\}^{|E|}$, where each position of the bitstring corresponds to an element from $E = \{e_1, \ldots, e_{|E|}\}$. A search point $s \in \{0,1\}^{|E|}$ corresponds to the subset $E(s) := \{e_i \in E \mid s_i = 1, 1 \leq i \leq |E|\}$ of E. We define the weight $w(s)$ of a bitstring s as the weight of the corresponding set $E(s)$. In a similar way we define the rank $r(s)$ of a bitstring s as $r(s) := r(E(s))$.

We consider two fitness functions f and f'. Let $w_{\max}$ denote the maximum weight of any element in E. Then $w_{ub} := |E| \cdot w_{\max}$ is an upper bound on the weight of any subset of E. Let

$$f(s) := (r(E) - r(s)) \cdot |E| \cdot w_{ub} + (|E(s)| - r(E)) \cdot w_{ub} + \sum_{e \in E(s)} w(e)$$

be the first fitness function which is to be minimized. The expression is dominated by the first term which encodes the number $r(E) - r(s)$ of elements that have to be added to $E(s)$ to get a superset of a basis. If $E(s)$ is a superset of a basis, this term vanishes and the second term dominates the expression. The factor $|E(s)| - r(E)$ denotes the number of elements that have to be removed from $E(s)$ to get a basis of M. If $E(s)$ is a basis of M, both the first and second term vanish and the fitness of s corresponds to the weight of the elements in $E(s)$.

The second term in the fitness function f explicitly penalizes circuits. This is not necessary, since the removal of an element of a circuit leads to a weight decrease by itself. Therefore, we also investigate the fitness function

$$f'(s) := (r(E) - r(s)) \cdot w_{ub} + \sum_{e \in E(s)} w(e).$$

Note that $f'(s)$ equals $f(s)$ if s describes a spanning tree. We shall see that the additional information concerning $|E(s)|$ in the first fitness function f allows to obtain better bounds.

The remainder of this section is structured as follows. First we prove two propositions concerning the expected number of generations until a superset of a basis and a basis have been constructed. Then we present several propositions about basis transition properties which are needed to prove the main theorems of this section.

PROPOSITION 3.1. *The expected number of generations until RLS or (1+1) EA working on one of the fitness functions f or f' constructs a superset of a basis of M is bounded by $O(|E| \log r(E))$.*

PROOF. Suppose the initial search point s does not describe a superset of a basis. Then $r(s) < r(E)$ holds. Both fitness functions f and f' are defined in such a way that the rank of $E(s)$ will never decrease in accepted steps. For each subset $X \subseteq E$, there are at least $r(E) - r(X)$ elements of E whose inclusion increases the rank of X by 1.

The probability that a step increases the rank of $E(s)$ is at least $\frac{1}{2} \cdot \frac{r(E)-r(s)}{|E|}$ for RLS and $\frac{1}{2e} \cdot \frac{r(E)-r(s)}{|E|}$ for (1+1) EA. The latter probability is a lower bound on

$$(1 - |E|^{-1})^{r(s)} \cdot (1 - (1 - |E|^{-1})^{r(E)-r(s)})$$

which denotes the probability that $r(s)$ elements of a fixed basis of $E(s)$ remain unchanged and at least one of $r(E) - r(s)$ elements that enlarge this independent subset is flipped. Hence, the expected number of generations until s describes a superset of some basis is bounded from above by

$$\sum_{i=0}^{r(E)-1} \frac{2e|E|}{r(E) - i} = O(|E| \log r(E)).$$

This concludes the proof. □

PROPOSITION 3.2. *The expected number of generations until RLS or (1+1) EA working on the fitness function f constructs a basis of M starting from a superset of a basis is bounded by $O(|E| \log |E|)$.*

PROOF. Suppose the initial search points s describes a proper superset of some basis of M. Then $|E(s)| > r(E)$ holds. The fitness function f is defined in such a way that only supersets of bases are accepted, i.e., the rank of $E(s)$ does not change. Furthermore, the cardinality of $E(s)$ never increases.

The probability that a step decreases the cardinality of $E(s)$ while maintaining a superset of a basis is at least $\frac{1}{2} \cdot \frac{|E(s)|-r(E)}{|E|}$ for RLS and $\frac{1}{2e} \cdot \frac{|E(s)|-r(E)}{|E|}$ for (1+1) EA. The latter probability is a lower bound on

$$(1 - |E|^{-1})^{r(E)} \cdot (1 - (1 - |E|^{-1})^{|E(s)|-r(E)})$$

which denotes the probability that $r(E)$ elements of a fixed basis of M contained in $E(s)$ remain unchanged and at least one of the remaining $|E(s)| - r(E)$ elements is flipped. Hence, the expected number of generations until s describes a basis of M is bounded from above by

$$\sum_{i=r(E)+1}^{|E|} \frac{2e|E|}{i - r(E)} = O(|E| \log |E|).$$

This concludes the proof. □

We remark that Proposition 3.2 holds also for RLS and the fitness function f', but not for (1+1) EA and f'. Since RLS flips at most two bits per step, an increase in $|E(s)|$ implies an increasing weight. Consequently, steps increasing $|E(s)|$ are not accepted. This argument does not hold for the (1+1) EA, which might exclude a heavy element and include two or more light elements instead while maintaining a superset of a basis.

The following proposition will later turn out to be useful in order to prove a bound on the number of steps needed to get from an arbitrary basis to a minimum weight basis.

PROPOSITION 3.3. *Given a matroid $M = (E, \mathcal{F})$ and two sets $A, B \in \mathcal{F}$ such that $C(B, a) \neq \emptyset$ for all $a \in A \setminus B$ holds. Then there exists a bipartite matching in the graph $G := (V_G, E_G)$, $V_G := A \triangle B$, $E_G := \{(a, b) \mid a \in A \setminus B, b \in C(B, a) \setminus A\}$ covering $A \setminus B$.*

PROOF. Note that G is a bipartite graph with bipartition $V = (A \setminus B) \dot\cup (B \setminus A)$. For $X \subseteq A \setminus B$ define $N(X) := \{b \in B \setminus A \mid \exists x \in X : (x, b) \in E_G\}$. We show $|N(X)| \geq |X|$ for all $X \subseteq A \setminus B$. Then the claim follows by the classical Theorem of Hall (see, e.g., [14]).

Suppose there exists $X \subseteq A \setminus B$ with $|N(X)| < |X|$. The sets X and $N(X)$ are independent and disjoint as subsets of $A \setminus B$ and $B \setminus A$, respectively. Moreover, the sets $X \dot\cup (A \cap B)$ and $N(X) \dot\cup (A \cap B)$ are independent as subsets of A and B, respectively, and $|N(X) \dot\cup (A \cap B)| < |X \dot\cup (A \cap B)|$ holds. Hence, by Definition 2.1, there exists $x \in (X \dot\cup (A \cap B)) \setminus (N(X) \dot\cup (A \cap B)) = X \setminus N(X) = X$ such that $N(X) \dot\cup (A \cap B) \dot\cup \{x\} \in \mathcal{F}$. On the other hand, by the definition of $N(\cdot)$, the set $N(X) \dot\cup (A \cap B) \dot\cup \{x\}$ contains the cycle $C(B, x)$. □

PROPOSITION 3.4. *Let s be a search point describing a non-minimum weight basis B of M. Then there exists some $k \in \{1, \ldots, r(E)\}$ and k different accepted 2-bit flips such that the average weight decrease of these flips is $(w(s) - w_{OPT})/k$.*

PROOF. Let B^* denote a minimum weight basis of M and define $k := |B^* \setminus B|$. By applying Proposition 3.3 to B and B^* we obtain an injective function $\alpha : B^* \setminus B \to B \setminus B^*$ such that $\alpha(e) \in C(B, e)$. By Proposition 2.2(i), all bases of a matroid have equal cardinality, hence, the function α is a bijection.

By Proposition 2.2(ii), $B \cup \{e\} \setminus \{\alpha(e)\}$ is again a basis of M. Furthermore, since B^* is an optimal basis $w(e) \leq w(\alpha(e))$ holds for all $e \in B^* \setminus B$. Hence, exchanging e and $\alpha(e)$ does not increase the total weight and the 2-bit flip involving e and $\alpha(e)$ is accepted. All k 2-bit flips together change B into B^* and the total weight decrease is $w(s) - w_{OPT}$. Hence, the average weight decrease is $(w(s) - w_{OPT})/k$. □

The analysis performed later can be simplified if the parameter k in Proposition 3.4 is independent of the search point s. This can be easily accomplished by allowing non-accepted 2-bit flips whose weight decrease is defined as 0. We add $r(E) - k$ non-accepted 2-bit flips to the k 2-bit flips from Proposition 3.4.

PROPOSITION 3.5. *Let s be a search point describing a basis B of M. Then there exists a set of $r(E)$ 2-bit flips such that the average weight decrease of these flips is $(w(s) - w_{OPT})/r(E)$.*

Since Proposition 3.2 does not hold for the fitness function f' in combination with (1+1) EA, we need a result similar to Proposition 3.5 for supersets of a basis. Since we start from supersets of a basis, we need to allow also 1-bit flips to reach a basis.

PROPOSITION 3.6. *Let s be a search point describing a superset of a basis. Then there exists a set of $|E| - r(E)$ 1-bit flips and a set of $r(E)$ 2-bit flips such that the average weight decrease is $(w(s) - w_{OPT})/|E|$.*

PROOF. Let $B' \subseteq E(s)$ denote a basis of M and s' the corresponding search point. Consider the set of $|E(s)| - r(E)$ 1-bit flips corresponding to the elements in $E(s) \setminus B'$. Their removal from $E(s)$ does not change the rank of $E(s)$, hence the 1-bit flips are accepted. We obtain the basis B' and apply Proposition 3.5. Alltogether, we obtain a weight decrease of $w(s) - w_{OPT}$ and performed $|E(s)|$ flips.

Similar to Proposition 3.5 we allow non-accepted 1-bit flips whose weight decrease is defined as 0. By adding $|E| - |E(s)|$ non-accepted 1-bit flips we obtain the claimed result. □

THEOREM 3.7. *The expected number of generations until RLS or (1+1) EA working on the fitness function f constructs a minimum weight basis is bounded by $O(|E|^2(\log r(E) + \log w_{\max}))$.*

PROOF. By Proposition 3.1 and 3.2, it is sufficient to consider the search process after having found a search point s describing a basis. Then, by Proposition 3.5, there exists a set of $r(E)$ 2-bit flips whose average weight decrease is $(w(s) - w_{OPT})/r(E)$. The choice of such a 2-bit flip is called a *good step*. The probability of performing such a good step equals $\Theta(r(E)/|E|^2)$ and each of the good steps is chosen with the same probability. A good step decreases the difference between the weight of the current search point s and w_{OPT} on average by a factor of $1 - 1/r(E)$. This holds independently of previous good steps. Hence, after N good steps, the expected difference between $w(s)$ and w_{OPT} is given by $(1 - 1/r(E))^N \cdot (w(s) - w_{OPT})$. Since $w(s) \leq r(E) \cdot w_{\max}$ and $w_{OPT} \geq 0$, we obtain the upper bound $(1 - 1/r(E))^N \cdot D$, where $D := r(E) \cdot w_{\max}$.

If $N := \lceil (\ln 2) \cdot r(E) \cdot (\log 2D) \rceil$, this bound is at most $\frac{1}{2}$. Since the difference is not negative, by Markov's inequality, the probability that the bound is less than 1 is at least 1/2. The difference is an integer implying that the probability of having found a minimum weight basis is at least 1/2. Therefore, the expected number of good steps until a minimum weight basis is found is bounded by $2N = O(r(E) \log D) = O(r(E)(\log r(E) + \log w_{\max}))$.

By our construction, there are always exactly $r(E)$ good steps. Therefore, the probability of a good step does not depend on the current search point. Hence, the expected number of generations until l good steps have been made equals $\Theta(l|E|^2/r(E))$. Altogether, the expected number of iterations is bounded by

$$O(N|E|^2/r(E)) = O(|E|^2(\log r(E) + \log w_{\max})).$$

This concludes the proof. □

A slightly worse bound can be shown for the fitness function f' by applying Proposition 3.6 instead of Proposition 3.5.

THEOREM 3.8. *The expected number of generations until RLS or (1+1) EA working on the fitness function f' constructs a minimum weight basis is bounded by* $O(|E|^2(\log|E| + \log w_{\max}))$.

PROOF. By Proposition 3.1, it is sufficient to consider the search process after having found a search point s describing a superset of a basis. Then, by Proposition 3.6, for each step there exists a set of $|E|-r(E)$ 1-bit flips and a set of $r(E)$ 2-bit flips such that the total weight decrease is $w(s) - w_{OPT}$. The choice of such a bit flip is called a *good step*. If the total weight decrease of the 1-bit flips is larger than the total weight decrease of the 2-bit flips, the step is called a 1-step. Otherwise, it is called a 2-step. Note that the notion of a 2-step does not imply that we actually perform a 2-bit flip, similarly for 1-steps and 1-bit flips.

Consider the sequence of all steps until a minimum weight basis is reached. Suppose that at least half of the required steps are 2-steps. Consider only these 2-steps. The probability of a good 2-bit flip equals $\Theta(r(E)/|E|^2)$. The expected weight decrease of such a 2-bit flip in a 2-step is at least $\frac{1}{2}(w(s) - w_{OPT})/r(E)$, resulting in a factor not larger than $1-1/(2r(E))$. Hence, we can adapt the proof of Theorem 3.7 with $N' := \lceil(\ln 2) \cdot 2r(E) \cdot (\log 2D')\rceil$, $D' := |E| \cdot w_{\max}$ and obtain the bound $O(N'|E|^2/r(E))$ for the expected number of 2-steps. Since the majority of all steps are 2-steps, the claimed result follows.

Now suppose that at least half of the required steps are 1-steps and consider only these 1-steps. The probability of a good 1-bit flip equals $\Theta(k/|E|)$ for $k = |E| - r(E)$. The expected weight decrease of such a 1-bit flip in a 1-step is at least $\frac{1}{2}(w(s) - w_{OPT})/k$, resulting in a factor not larger than $1 - 1/(2k)$. Again, we can apply the proof technique of Theorem 3.7 where $N'' := \lceil 2 \cdot (\ln 2) \cdot k \cdot (\log 2D')\rceil = O(|E|(\log|E| + \log w_{\max}))$ takes the role of N. We obtain the upper bound

$$O(N''|E|/k) = O(|E|(\log|E| + \log w_{\max}))$$

for the expected number of 1-steps. Since the majority of all steps are 1-steps, the total number of steps is of the same order, which is even smaller than the proposed bound. □

NEUMANN and WEGENER [16] show a lower bound of $\Omega(|E|^2 \log r(E))$ for a special class of instances of the minimum spanning tree problem. Hence, $\Omega(|E|^2 \log r(E))$ is also a lower bound for the minimum weight basis problem.

We briefly want to mention the benefits of more problem-specific mutation operators. After having found a basis of a matroid, we are only interested in offspring with the same cardinality. The probability of such an offspring can be increased using the following two mutation operators: If RLS flips two bits, it chooses randomly a 0-bit and randomly a 1-bit. If s contains k 1-bits, (1+1) EA flips each 1-bit with probability $1/k$ and each 0-bit with probability $1/(|E| - k)$.

Using the modified mutation operators, the probability of a specific element exchange for bases increases from $\Theta(1/|E|^2)$ to $\Theta(r(E)^{-1}(|E|-r(E))^{-1})$. Therefore, the bound of Theorem 3.7 can be replaced by $O(r(E)|E|(\log r(E)+\log w_{\max})+|E|\log|E|)$. In the case of Theorem 3.8 we obtain the bound $O(r(E)|E|(\log|E| + \log w_{\max}))$.

The expected number of generations can be further reduced by using parallel versions of (1+1) EA and RLS that produce several offspring in each iteration. The (1+λ) EA and λ-PRLS algorithm produce independently λ offspring from the single individual of the current population. The selection procedure selects an individual with the smallest fitness value among the parent and its offspring. In the proofs of Theorem 3.7 and Theorem 3.8 the probability of a good step is $O(r(E)/|E|^2)$. Choosing $\lambda := \lceil|E|^2/r(E)\rceil$, this probability is increased to a positive constant. As before, the expected number of good steps is bounded by $O(r(E)(\log r(E)+\log w_{\max}))$ and $O(r(E)(\log|E|+\log w_{\max}))$, respectively. This leads to the following result.

THEOREM 3.9. *The expected number of generations until λ-PRLS or (1+λ) EA with $\lambda := \lceil|E|^2/r(E)\rceil$ children constructs a minimum weight basis is bounded by* $O(r(E)\log w_{\max} + |E|\log|E|)$.

Using the modified mutation operator mentioned above, the probability of a good step reduces to $O(1/|E|)$ and we obtain the same bound on the expected number of generations as in Theorem 3.9 already for $\lambda := |E|$.

NEUMANN and WEGENER [17] also consider multi-objective optimization techniques for the (single-objective) minimum spanning tree problem. They study the behavior of two multi-objective evolutionary algorithms called SEMO and GSEMO. The analysis can be carried over to the minimum weight basis problem for matroids and results in a bound of $O(r(E)|E|(r(E)+\log|E|+\log w_{\max}))$ on the number of generations.

4. MATROID INTERSECTION

The results presented in this section are motivated by the results for the maximum matching problem in [9]. We consider the *matroid intersection problem* which is defined as follows. Given two matroids $M_1 = (E, \mathcal{F}_1)$ and $M_2 = (E, \mathcal{F}_2)$ on the same ground set E by their independence oracles, compute a set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that $|X|$ is maximum. Let OPT denote such an optimal element of $\mathcal{F}_1 \cap \mathcal{F}_2$. The well-known matroid intersection algorithm by EDMONDS [3, 14] starts with $X := \emptyset$. In each iteration, it searches a shortest S_X-T_X-path in the auxiliary graph G_X. This so-called *augmenting path* gives rise to $X' \in \mathcal{F}_1 \cap \mathcal{F}_2$ with $|X'| = |X| + 1$. The algorithm terminates if there is no augmenting path.

The auxiliary graph G_X for $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ is defined as follows. Its node set is E, the edges are given by $A_X \cup B_X$ with $A_X := \{(x, y) \mid y \in E \setminus X, x \in C_1(X, y) \setminus \{y\}\}$ and $B_X := \{(y, x) \mid y \in E \setminus X, x \in C_2(X, y) \setminus \{y\}\}$. We set $S_X := \{y \in E \setminus X \mid X \cup \{y\} \in \mathcal{F}_1\}$ and $T_X := \{y \in E \setminus X \mid X \cup \{y\} \in \mathcal{F}_2\}$.

Let the node sequence $y_0, x_1, y_1, \ldots, x_n, y_n$ denote any shortest S_X-T_X-path and define $X' := X \setminus \{x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_n\}$. Then this path is an augmenting path, i.e., $X' \in \mathcal{F}_1 \cup \mathcal{F}_2$ and $|X'| = |X| + 1$. In the EA setting, such an augmentation step corresponds to simultaneously flipping exactly the elements corresponding to the nodes of the augmenting path.

The above algorithm solves the matroid intersection problem in $O(|E|^3\theta)$ time, where θ is the maximum complexity of both independence oracles. Faster matroid intersection algorithms are due to CUNNINGHAM [2] and GABOW and XU [6].

We study the performance of evolutionary algorithms for the matroid intersection problem. We assume that we are given rank oracles r_1 and r_2 that compute for any set $X \subseteq E$

its rank with respect to M_1 and M_2, respectively. Again, we consider the RLS and (1+1) EA algorithm. We consider the fitness function

$$f(s) := -\Phi(s) \cdot |E| + |E(s)|,$$

where $\Phi(s) := 2|E(s)| - r_1(E(s)) - r_2(E(s))$. The expression is dominated by $\Phi(s)$, which measures the infeasibility of $E(s)$. If $E(s)$ is a common independent set, this first term vanishes and the fitness of s equals the cardinality of $E(s)$.

A more precise way to measure the infeasibility of $E(s)$ is to replace $\Phi(s)$ by $\Psi(s) := \min\{|X| | X \subseteq E, E(s) \setminus X \in \mathcal{F}_1 \cap \mathcal{F}_2\}|$. However, $\Psi(s)$ cannot be easily computed. Hence, we resort to $\Phi(s)$. Note that $\frac{1}{2}\Phi(s) \leq \Psi(s) \leq \Phi(s) \leq 2|E|$ holds for all $s \in \{0,1\}^{|E|}$.

First, we consider the phase until a common independent set has been constructed. Note that the empty set is a trivial common independent set. Hence, the first phase can also be skipped entirely.

Proposition 4.1. *The expected number of generations until RLS or (1+1) EA working on the fitness function f constructs a common independent set is bounded by $O(|E| \log |E|)$.*

Proof. Suppose $E(s) \notin \mathcal{F}_1 \cap \mathcal{F}_2$ holds for the initial search point s. The fitness function f is defined in such a way that the infeasibility $\Phi(s)$ never increases. There are at least $\frac{1}{2}\Phi(s)$ elements in $E(s)$ that lead to a decrease of the infeasibility $\Phi(s)$. The probability that a step decreases the infeasibility $\Phi(s)$ is at least $\frac{1}{2} \cdot \frac{\Phi(s)}{2|E|}$ for RLS and at least $\frac{1}{2e} \cdot \frac{\Phi(s)}{2|E|}$ for (1+1) EA. The last probability is a lower bound for $(1 - |E|^{-1})^{|E \setminus E(s)|} \cdot (1 - (1 - |E|^{-1})^{\Phi(s)/2})$, which denotes the probability that the complement of $E(s)$ remains fixed and at least one of $\frac{1}{2}\Phi(s)$ infeasible elements is removed from $E(s)$. Hence, the expected number of generations until s describes an element in $\mathcal{F}_1 \cap \mathcal{F}_2$ is bounded from above by

$$\sum_{i=1}^{\Phi(s)} \frac{4e|E|}{i} = O(|E| \log |E|).$$

This concludes the proof. □

Next, we consider the search process after having found a common independent set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$. We show that the length of a shortest S_X-T_X-path in G_X can be bounded in terms of $|X|$ and $|OPT|$.

Proposition 4.2. *Let $\epsilon > 0$ and $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that $|X| < (1-\epsilon)|OPT|$. There exists an S_X-T_X-path in G_X with length at most $2\lceil 1/\epsilon \rceil - 2$.*

Proof. It was shown by Cunningham [2] that G_X contains $k := |OPT| - |X|$ disjoint S_X-T_X-paths. Hence, there exists an S_X-T_X-path with length at most $2\lfloor |X|/k \rfloor$. Since $|X| < (1-\epsilon)\,|OPT|$, we have $|X|/k < (1-\epsilon)/\epsilon < 1/\epsilon$. Thus, the length of this path is bounded by $2\lceil 1/\epsilon \rceil - 2$. □

The bound on the length of an augmenting path allows us to lower bound the probability that RLS or (1+1) EA finds such an augmenting path. These bounds lead to upper bounds on the expected number of generations until $|E(s)|$ is increased, and finally, until an $(1-\epsilon)$-approximation is constructed.

Theorem 4.3. *For $\epsilon > 0$, the expected number of generations until RLS or (1+1) EA working on the fitness function f constructs an $(1-\epsilon)$-approximation of a maximum element of $\mathcal{F}_1 \cap \mathcal{F}_2$ is bounded by $O(|E|^{2\lceil 1/\epsilon \rceil})$.*

Proof. By Proposition 4.1, it suffices to consider the search process after having found a search point s with $E(s) \in \mathcal{F}_1 \cap \mathcal{F}_2$. The fitness function f is designed such that only steps leading to search points s' describing common independent sets of at least the same cardinality as s are accepted. Assume that $|E(s)| < (1-\epsilon)|OPT|$.

By Proposition 4.2, there exists an augmenting path in $G_{E(s)}$ of length at most $l := 2\lceil 1/\epsilon \rceil - 2$. The (1+1) EA flips exactly the $l+1$ elements corresponding to the nodes of this path with probability $\Omega(|E|^{-l-1})$. The RLS algorithm needs $l/2$ 2-bit flips shortening the augmenting path and a final 1-bit flip to increase $|E(s)|$. The probability that this happens within the next $l/2+1$ steps is bounded from below by $\Omega((|E|^{-2})^{l/2} \cdot |E|^{-1}) = \Omega(|E|^{-l-1})$. Hence, the expected number of generations to improve $|E(s)|$ is bounded by $O(|E|^{l+1})$ for (1+1) EA and by $O(l \cdot |E|^{l+1})$ for RLS. A more careful analysis for RLS yields the bound $O(|E|^{l+1})$ (see [9]). Since $|OPT| \leq |E|$, the expected number of generations until RLS or (1+1) EA constructs an $(1-\epsilon)$-approximation of a maximum element of $\mathcal{F}_1 \cap \mathcal{F}_2$ is bounded by $O(|E| \log |E|) + O(|E|) \cdot O(|E|^{l+1}) = O(|E|^{2\lceil 1/\epsilon \rceil})$. □

Giel and Wegener [9] have shown that RLS and (1+1) EA require an exponential expected running time to find an optimal solution for certain bipartite maximum matching problems. Since bipartite matching is a special case of matroid intersection, we know that the matroid intersection problem cannot be solved by RLS or (1+1) EA in polynomial time.

5. WEIGHTED MATROID INTERSECTION

In the *weighted matroid intersection problem* we additionally consider a weight function $w : E \to \mathbb{N}$ which assigns a non-negative weight $w(e)$ to each element $e \in E$ of the ground set. The task is to compute a common independent set $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that its weight $w(X) := \sum_{e \in X} w(e)$ is maximum.

Similar to the unweighted case we consider the fitness function

$$f(s) := -\Phi(s) \cdot w_{ub} + \sum_{e \in E(s)} w(e),$$

where $w_{ub} := |E| \cdot w_{\max}$ is an upper bound on the weight of any subset of E.

Note that the RLS algorithm is not suited for the weighted matroid intersection problem since, in general, simultaneous flips of more than two bits are required. In the unweighted case, a long augmenting path can be broken into a series of 2-bit flips maintaining the fitness value and one final 1-bit flip. In the weighted case, there are simple examples of paths of length five where such a decomposition into a sequence of 2-bit flips with non-negative difference of the fitness value does not exist.

Although bit flips of at most two bits are not sufficient in the weighted case, it is possible to obtain a $\frac{1}{2}$-approximation using only bit flips of at most three bits. We restrict ourselves in the following to this special setting and analyze the expected number of generations to obtain a $\frac{1}{2}$-approximation.

PROPOSITION 5.1. *Let s be a search point such that its fitness value $f(s)$ cannot be improved by flipping at most three bits. Then $w(s) \geq \frac{1}{2} w_{OPT}$ holds.*

PROOF. Define $A := OPT$ and $B := E(s)$. Since the fitness value $f(s)$ cannot be improved by flipping one or two bits of s, we have $C_1(B, a) \neq \emptyset$ and $C_2(B, a) \neq \emptyset$ for all $a \in A \setminus B$. Define $X := A \setminus B = \{x_1, \ldots x_k\}$.

By Proposition 3.3, there exist sets $Y' = \{y'_1, \ldots, y'_k\} \subseteq B \setminus A$ and $Y'' = \{y''_1, \ldots, y''_k\} \subseteq B \setminus A$ such that $y'_i \in C_1(B, x_i)$ and $y''_i \in C_2(B, x_i)$ for $1 \leq i \leq k$. Since the fitness value $f(s)$ cannot be improved by flipping at most three bits of s, we have $w(x_i) \leq w(y'_i) + w(y''_i)$ for $1 \leq i \leq k$. Summing up these inequalities yield $w(A \setminus B) = w(X) \leq w(Y') + w(Y'') \leq 2\, w(B \setminus A)$. Hence, $w(s) \geq \frac{1}{2} w_{OPT}$ holds. □

In order to analyze the expected number of generations until a $\frac{1}{2}$-approximate solution is found we prove that there exists at least one bit flip with a certain weight increase.

PROPOSITION 5.2. *Let s be a search point such that $w(s) \leq (\frac{1}{2} - \epsilon)\, w_{OPT}$ holds for some $\epsilon > 0$. Then there exists an accepted bit flip involving at most three bits with a weight increase of at least $\frac{2\epsilon}{|E|} w_{OPT}$.*

PROOF. Define $A := OPT$, $B := E(s)$ and $X := A \setminus B = \{x_1, \ldots, x_k\}$. We use the index sets I' and I'' to denote those elements of X whose addition to B causes a cycle in the matroid M_1 and M_2, respectively.

$$I' := \{i \mid 1 \leq i \leq k,\, C_1(B, x_i) \neq \emptyset\}$$
$$I'' := \{i \mid 1 \leq i \leq k,\, C_2(B, x_i) \neq \emptyset\}$$

By Proposition 3.3, there exist a set $Y' = \{y'_i \mid i \in I'\} \subseteq B \setminus A$ such that $y'_i \in C_1(B, x_i)$ for all $i \in I'$. Likewise, there exists a set $Y'' = \{y''_i \mid i \in I''\} \subseteq B \setminus A$ such that $y''_i \in C_2(B, x_i)$ for all $i \in I''$. We define the weights w_i, w'_i and w''_i for $1 \leq i \leq k$ as follows:

$$w_i := w(x_i)$$
$$w'_i := \begin{cases} w(y'_i) & \text{if } i \in I' \\ 0 & \text{otherwise} \end{cases}$$
$$w''_i := \begin{cases} w(y''_i) & \text{if } i \in I'' \\ 0 & \text{otherwise} \end{cases}$$

By assumption, $w(B) \leq (\frac{1}{2} - \epsilon)\, w(A)$ holds. Hence, we have $w(A \setminus B) - w(B \setminus A) \geq (\frac{1}{2} + \epsilon)\, w(A)$. Since $w(B \setminus A) \leq w(B) \leq (\frac{1}{2} - \epsilon)\, w(A)$, it follows that $w(A \setminus B) - 2w(B \setminus A) \geq 2\epsilon\, w(A)$. We have

$$\sum_{i=1}^{k} w_i - w'_i - w''_i = w(X) - w(Y') - w(Y'')$$
$$\geq w(A \setminus B) - 2w(B \setminus A) \geq 2\epsilon\, w(A)$$

Hence, there exists an $i \in \{1, \ldots, k\}$ such that $w_i - w'_i - w''_i \geq \frac{2\epsilon}{k} w(A)$. Consider the bit flip that adds the element x_i and removes the elements y'_i and y''_i if $i \in I'$ and $i \in I''$, respectively (note that y'_i and y''_i might be identical). This bit flip involves at most three bits and has a weight increase of at least $\frac{2\epsilon}{|E|} w(A)$. By construction, the resulting bit string encodes a common independent set and the bit flip is accepted. □

Now we can prove our main result, the expected number of generations for a $\frac{1}{2}$-approximation of the weighted matroid intersection problem.

THEOREM 5.3. *The expected number of generations until $(1+1)$ EA working on the fitness function f constructs a $\frac{1}{2}$-approximation of a maximum weight element of $\mathcal{F}_1 \cap \mathcal{F}_2$ is bounded by $O(|E|^4(\log r + \log w_{\max}))$, where $r := \min\{r_1(E), r_2(E)\}$.*

PROOF. By Proposition 4.1 (which also holds for the weighted case), it suffices to consider the search process after having found a search point s with $E(s) \in \mathcal{F}_1 \cap \mathcal{F}_2$. The fitness function f is designed such that only steps leading to search points s' that describe common independent sets of at least the same weight as s are accepted.

Now consider any search point s with $E(s) \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $w(s) < \frac{1}{2} w_{OPT}$. Define $\epsilon := \frac{1}{2} - \frac{w(s)}{w_{OPT}}$, i.e., $w(s) = (\frac{1}{2} - \epsilon)\, w_{OPT}$ holds. By Proposition 5.2 there exists an accepted bit flip involving at most three bits with a weight increase of at least $\frac{2\epsilon}{|E|} w_{OPT}$. Such a step is called a *good step*. A good step decreases the difference $\epsilon \cdot w_{OPT}$ between the weight $w(s)$ of the current search point s and $\frac{1}{2} w_{OPT}$ by a factor not larger than $1 - 2/|E|$. Hence, after N good steps, the difference between $w(s)$ and $\frac{1}{2} w_{OPT}$ is bounded from above by $(1 - 2/|E|)^N \cdot (\frac{1}{2} w_{OPT} - w(s))$. Since $w_{OPT} \leq r \cdot w_{\max}$ and $w(s) \geq 0$, we obtain the upper bound $(1 - 2/|E|)^N \cdot D$, where $D := \frac{1}{2} r \cdot w_{\max}$.

If $N := \lceil (\ln 2) \cdot \frac{|E|}{2} \cdot \log(3D) \rceil$, this bound is at most $\frac{1}{3}$. The difference is half-integral which implies that we have actually reached a $\frac{1}{2}$-approximation after at most N good steps. The probability of a good step is bounded from below by $\Omega(|E|^{-3})$. Hence, the expected number of generations for N good steps is bounded by

$$O(N|E|^3) = O(|E|^4(\log r + \log w_{\max})).$$

This concludes the proof. □

Consider the following modification of the RLS algorithm. Choose $b \in \{0, 1, 2\}$ randomly. If $b < 2$ proceed as before. Otherwise, choose $(i, j, k) \in \{(a, b, c) \mid 1 \leq a < b < c \leq |E|\}$ randomly and flip the i-th, j-th and k-th bit of s. We call this algorithm RLS3.

Since we restrict ourselves to bit flips involving at most three bits, all good steps that are accepted by the (1+1) EA can also be achieved using RLS3. Moreover, the probability of a particular bit flip is again bounded from below by $\Omega(|E|^{-3})$. Hence, Theorem 5.3 does not only hold for (1+1) EA, but also for RLS3.

6. INTERSECTION OF THREE OR MORE MATROIDS

Furthermore, the result of Theorem 5.3 can be easily generalized to the intersection of p matroids $M_i = (E, \mathcal{F}_i)$, $1 \leq i \leq p$. The task is to compute an independent set $X \in \bigcap_{i=1}^{p} \mathcal{F}_i$ with maximum weight. This problem is NP-hard for $p \geq 3$, as finding a Hamiltonian circuit in a directed graph is a special case; see [13].

Similar to the previous case of $p = 2$, there are situations in which simultaneous flips of at least $p+1$ bits are required. Therefore, we do not consider the RLS algorithm in this section. A modification of the RLS algorithm similar to that

described in the last paragraphs of the preceding section is still possible though.

The definition of the function $\Phi(s)$ is adjusted in the obvious way. The bound of Proposition 4.1 increases to $O(p|E|\log|E|)$. The results of Proposition 5.2 carry over to the intersection of p matroids, although the achieved approximation ratio is worse.

PROPOSITION 6.1. *Let s be a search point such that $w(s) \leq (\frac{1}{p} - \epsilon)\, w_{OPT}$ holds for some $\epsilon > 0$. Then there exists an accepted bit flip involving at most $p+1$ bits with a weight increase of at least $\frac{p\,\epsilon}{|E|} w_{OPT}$.*

The lower bound for the probability of picking a particular bit flip of at most $p+1$ bits reduces to $\Omega(|E|^{-p-1})$. This observation leads to the following generalization of Theorem 5.3.

THEOREM 6.2. *Given p matroids $M_i = (E, \mathcal{F}_i)$, $1 \leq i \leq p$, the expected number of generations until (1+1) EA working on the fitness function f constructs a $\frac{1}{p}$-approximation of a maximum weight element of $\bigcap_{1 \leq i \leq p} \mathcal{F}_i$ is bounded by $O(|E|^{p+2}(\log r + \log w_{\max}))$, where $r := \min\{r_i(E) \mid 1 \leq i \leq p\}$.*

Similar to the minimum weight basis problem we can use parallel versions of (1+1) EA and RLS to reduce the number of generations. Choosing the number of offspring per generation as $\lambda := |E|^{p+1}$ improves the probability of a good step from $\Omega(|E|^{-p-1})$ to a positive constant. As before, the expected number of good steps is bounded by $O(|E|(\log r + \log w_{\max}))$. This leads to the following result.

COROLLARY 6.3. *Given p matroids $M_i = (E, \mathcal{F}_i)$, $1 \leq i \leq p$, the expected number of generations until (1+λ) EA with $\lambda := |E|^{p+1}$ children working on the fitness function f constructs a $\frac{1}{p}$-approximation of a maximum weight element of $\bigcap_{1 \leq i \leq p} \mathcal{F}_i$ is bounded by $O(|E|(\log r + \log w_{\max}))$.*

7. CONCLUSION

We have analyzed the performance of (1+1) EA and RLS on a very general class of combinatorial optimization problems ranging from very simple problems that can be solved optimally by the greedy method up to NP-hard problems. Our results provide an indication of the enormous power of evolutionary algorithms from a theoretical point of view. It turns out that the very general and abstract structure of matroid optimization problems suffices to lead evolutionary algorithms into promising directions and to finally obtain optimal or at least provably good solutions after only polynomially many iterations.

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